Optimal Rigid-Body Motions

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Optimal rigid-body angular motions are investigated in the absence of direct control over one of the angular velocity components. A numerical survey of first-order necessary conditions for optimality reveals that, over a range of boundary conditions, there are, in general, several distinct extremal solutions. A classification in terms of subfamilies of extremal solutions is presented. Domains of existence of the extremal subfamilies are established. Locus of Darboux points are obtained, and global optimality of extremal solutions is observed in relation to Darboux points. Local optimality for the candidate minimizers is verified by investigating the second-order necessary conditions.

I. Introduction

STUDIES of optimal flight generally employ dynamic models based on point-mass equations of motion. Indeed, for many purposes these models must be further simplified, leading to energy models and the like. Recently there has been considerable interest in highly agile or supermaneuverable aircraft. Maneuvers of interest in these applications include rapid fuselage pointing and suggest the need to include rigid-body rotational dynamics in optimal flight studies. The usual flight mechanics wisdom is that the rigid-body rotations are "fast" compared with translational motions (short-period vis-à-vis phugoid frequencies). This suggests a singular perturbation scheme to divide and conquer these problems. However, some of the maneuvers discovered in simulator studies are delicately choreographed so that considerable care is needed in choice of coordinates and in time-scale selection. The key point is that careful analysis is needed to sort out various effects and to gain insight into these "supermaneuvers."

The present study is motivated by the problem of angular velocity control for a supermaneuverable aircraft. In one scenario of interest, primarily the low-dynamic-pressure regime at high angles of attack, the usual aerodynamic control surfaces are ineffective for generating control moments. In such flight regimes, it is proposed that thrust-vectored propulsive controls be used as the primary source of control moments, which may be generated by deflecting vanes in the exhaust flowfield of a jet-powered aircraft. Owing to the location of engines, we anticipate that the rolling moment (in the body-fixed axes) that can be generated is going to be small in comparison to the pitching and the yawing moments. As an extreme case, we consider the case when the rolling moment is identically zero. Hence any rolling motion is controlled indirectly through the pitch and yaw motions. Thus we are motivated to study optimal rigid-body rotational maneuvers in the absence of rolling moments.

Received May 1, 1989; presented as Paper 89-3616 at the AIAA Guidance, Navigation, and Control Conference, Boston, MA, Aug. 14–16, 1989; revision received Oct. 2, 1989.

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Optimal rigid-body rotational maneuvers were first studied in the context of spacecraft attitude control systems. The rotational dynamics about the center of mass of a rigid body are governed by the familiar Euler equations. These governing equations have quadratic nonlinearities due to gyroscopic cross-coupling terms. Athans and Falb[1] investigated the time-optimal and "fuel-optimal" angular velocity control of an axisymmetric rigid body. In Refs. 1 and 2, a time-optimal feedback law has been synthesized for regulating the angular momentum of an asymmetric rigid body, under special controller constraints. Optimal linear feedback laws are obtained by Debs and Athans[3] and Dwyer[4] in closed form for regulating the angular momentum of an arbitrarily rotating asymmetric rigid body. In Refs. 1 and 2, a time-optimal feedback law has been synthesized for regulating the angular momentum of an asymmetric rigid body, under special controller constraints. Optional linear feedback laws are obtained by Debs and Athens[3] and Dwyer[4] in closed-form for regulating the angular momentum of an arbitrarily rotating asymmetric rigid body. The linear time-invariant feedback law is optimal with respect to a quadratic cost functional in angular momenta and control torques. Further, in Ref. 3 nonlinear feedback laws are obtained that are optimal with respect to some nonquadratic cost functionals. In a more recent study, Golubev and Demidov[5] have investigated minimum-time and minimum-control effort rotation stoppage for an asymmetric spacecraft. The closed-form feedback laws obtained therein are applicable to a fairly general class of problems, with some restrictions on the actuator influence coefficients.

Optimal large angle attitude maneuvers for a rigid asymmetric vehicle are investigated by Junkins and Turner.[6] A relaxation process is presented to obtain solutions of arbitrary boundary conditions starting with analytical solutions for single axis maneuvers. Carrington and Junkins[7] obtained a polynomial feedback law for large angle attitude maneuvers. Minimum-time attitude slew maneuvers of a rigid spacecraft are obtained by Li and Bainum.[8] In all these studies, the usual first-order necessary conditions are used to construct extremal solutions. The present paper deals with optimal control of the angular velocity of the rigid body and not the more complex problem of attitude control. On the other hand, we take some care to investigate all extremal families and to verify (local) optimality. Additionally, in the existing literature, the interesting possibility of controlling arbitrary angular motion in the absence of direct control over one of the angular velocity components is not explored. The underlying assumption in the previous studies is that the control torquers have direct influence over each of the three components of the angular velocity.

Optimal rigid-body rotational maneuvers with pitch and yaw controllers are investigated via an approximate dynamic model in Ref. 9. The approximate dynamic model consists of two simple integrators representing the pitch and yaw dynamics and a bilinear isoperimetric constraint representing the roll dynamics. The approximate problem admits analytical solutions of the first-order necessary conditions for optimality and reveals an elaborate extremal structure. In this paper we corroborate and extend the findings of Ref. 9 using the exact dynamic model. In the following sections, we define the considered optimal control problem and present an analysis of first- and second-order necessary conditions for optimality.

II. Problem Statement

Consider the rotational dynamics of a rigid body in a body-fixed coordinate system. The origin of the coordinate system is located at the center of mass of the body, and the axes are aligned along its principal moments of inertia directions. Let ω_x, ω_y, and ω_z be the components of angular velocity denoting the roll, pitch, and yaw rates, respectively. Assume the body is free of all external moments except the applied control moments. Then the Euler equations of motion describing the rotational dynamics of the rigid body are

$$\dot{\omega}_x = J_x\omega_y\omega_z + (u_x/I_x) \tag{1}$$

$$\dot{\omega}_y = J_y\omega_x\omega_z + (u_y/I_y) \tag{2}$$

$$\dot{\omega}_z = J_z\omega_x\omega_y + (u_z/I_z) \tag{3}$$

where

$$J_x = \frac{I_y - I_z}{I_x}, \qquad J_y = \frac{I_z - I_x}{I_y}, \qquad J_z = \frac{I_x - I_y}{I_z}$$

and where I_x, I_y, I_z are the principal moments of inertia, and u_x, u_y, and u_z are the rolling, pitching, and yawing moments, respectively. We shall consider the case when $u_x(t) \equiv 0$.

We seek to perform an arbitrary point-to-point maneuver (from a given fixed point in state space to another) in a given time t_f while minimizing a weighted integral norm on control, defined to be the control effort, as follows

$$\mathscr{I} = \frac{1}{2}\int_0^{t_f} [(\alpha_y u_y)^2 + (\alpha_z u_z)^2]\,\mathrm{d}t \tag{4}$$

where α_y and α_z are nonzero, control weighting parameters.

The class of minimum control effort problems for linear systems was formulated and discussed by Neustadt in Ref. 10.

Note that the parameters of the aforementioned optimal control problem are 1) boundary conditions, namely: $\omega_x(0)$, $\omega_y(0)$, $\omega_z(0)$, $\omega_x(t_f)$, $\omega_y(t_f)$, $\omega_z(t_f)$, and t_f; 2) inertia characteristics: I_x, I_y, and I_z; and 3) ratio of control weighting parameters: α_y/α_z.

In an effort to consolidate the ensuing results for a wide spectrum of applications, we consider the following scaled version of the previously stated problem. Assuming that $I_z > I_y > I_x$, which is typically true for combat aircraft configurations, we define the following transformation:

$$p = c_p\omega_x \qquad q = c_q\omega_y \qquad r = c_r\omega_z$$

$$M = (c_q/I_y)u_y \qquad N = (c_r/I_z)u_z \tag{5}$$

where

$$c_p = \sqrt{J_y' J_z'}, \qquad c_q = \sqrt{J_x' J_z'}, \qquad c_r = \sqrt{J_x' J_y'}$$

$$\text{and} \quad J_x' = -J_x, \quad J_y' = J_y, \quad \text{and} \quad J_z' = -J_z$$

The equations of motion [Eqs. (1–3)] are transformed to

$$\dot{p} = -qr \tag{6}$$

$$\dot{q} = pr + M \tag{7}$$

$$\dot{r} = -pq + N \tag{8}$$

For a particular choice of the ratio of control weighting parameters,

$$\alpha_y/\alpha_z = c_q I_z / c_r I_y$$

whereby we weight equally the control influence in the $\dot{q}$ and $\dot{r}$ equations, the control effort, Eq. (4), transforms to

$$\mathscr{I} = \frac{1}{2}\int_0^{t_f} [M^2 + N^2]\,\mathrm{d}t \tag{9}$$

The problem of control remains the same; i.e., we seek to perform a point-to-point maneuver in a given time t_f, subject to dynamics given by Eqs. (6–8), while minimizing the control effort given by Eq. (9). The parameters for this problem are the initial and final conditions of the states p, q, and r and the final time t_f. We wish to emphasize that the scaled problem retains the important ingredients of the unscaled problem while reducing the number of parameters to be considered in analysis. All subsequent analysis pertains to the scaled problem. For specific values of I_x, I_y, and I_z, solutions may be recovered using the transformation defined in Eq. (5).

III. Derivation of First-Order Necessary Conditions

For convenience, we analyze the previously stated problem for first-order necessary conditions for optimality in the Mayer formulation. In this formulation, the cost appears as an additional state c. We seek to minimize $c(t_f)$ for a given point-to-point maneuver in a prescribed time t_f. The state equations are

$$\dot{p} = -qr \tag{10}$$

$$\dot{q} = pr + M \tag{11}$$

$$\dot{r} = -pq + N \tag{12}$$

$$\dot{c} = ½(M^2 + N^2) \tag{13}$$

where the controls $M(\cdot)$, $N(\cdot)$ are real-valued functions with no prescribed bounds. The boundary conditions are prescribed to be

$$p(0) = p_0 \quad \text{and} \quad p(t_f) = p_f \tag{14}$$

$$q(0) = q_0 \quad \text{and} \quad q(t_f) = q_f \tag{15}$$

$$r(0) = r_0 \quad \text{and} \quad r(t_f) = r_f \tag{16}$$

$$c(0) = 0 \tag{17}$$

Following the conventional practice, we define the variational-Hamiltonian $\mathscr{H}$ as

$$\mathscr{H} = (\lambda_0/2)(M^2 + N^2) + \lambda_p(-qr) + \lambda_q(pr + M) + \lambda_r(-pq + N) \tag{18}$$

where λ_0 is the adjoint variable associated with the cost and λ_p, λ_q, and λ_r are the adjoint variables associated with the states p, q, and r, respectively.

Using a version of Pontryagin's minimum principle, which states that an optimal control (M^*, N^*) is one that minimizes the Hamiltonian $\mathscr{H}$ over all possible controls (M, N), we derive, quite simply, the extremal control (M^*, N^*) as

$$M^*(t) = -\lambda_q^*(t)/\lambda_0^*(t) \quad (19)$$

$$N^*(t) = -\lambda_r^*(t)/\lambda_0^*(t) \quad (20)$$

where the adjoint variables λ_0, λ_p, λ_q, and λ_r satisfy

$$\dot{\lambda}_0 = 0 \quad (21)$$

$$\dot{\lambda}_p = -\lambda_q r + \lambda_r q \quad (22)$$

$$\dot{\lambda}_q = -\lambda_p r + \lambda_r p \quad (23)$$

$$\dot{\lambda}_r = -\lambda_p q - \lambda_q p \quad (24)$$

If we assume normality, the transversality condition gives

$$\lambda_0(t_f) = 1 \quad (25)$$

and we obtain $\lambda_0(t) \equiv 1$ by integrating Eq. (21). Further, the problem is *regular* since the matrix of the second partial derivatives of $\mathscr{H}$ with respect to controls M, N is the identity matrix. Thus, the Legendre-Clebsch condition is strictly satisfied.

The state equations, [Eqs. (10-13)] and the adjoint equations [Eqs. (21-24)], along with extremal control equations [Eqs. (19) and (20)] and the boundary conditions defined by Eqs. (14-17) and Eq. (25), constitute a nonlinear, two-point boundary-value problem (TPBVP). Numerical solutions of the TPBVP are obtained iteratively using a multiple-shooting algorithm.[11] The initial guess values of the state-adjoint trajectory, required by the TPBVP solver, are obtained using the analytical solutions of the approximate problem presented in Ref. 9.

IV. Extremal Families

Recall that extremals are paths that satisfy the usual first-order necessary conditions. In the present case, they are the solutions of the TPBVP defined in the preceding section. Observe that the extremal solutions of this TPBVP are, in general, a seven-parameter family of solutions. Of the many choices for these parameters, one may choose for instance, p_0, q_0, r_0, p_f, q_f, r_f, and t_f as these seven parameters. Since the problem is nonlinear with split boundary conditions, uniqueness of the extremal solution is not guaranteed for specified values of the above parameter list. In this section we will show, via numerical examples, that for a given set of initial and final conditions, there are, in general, several distinct extremal solutions. Each extremal solution is then classified to belong to a distinct subfamily, based on the nature of the extremal control. Further, in a reduced parameter space we shall establish domains of existence of some of the extremal subfamilies and show that these domains overlap. In the overlapping regions, where more than one extremal solution exists, a locus of Darboux[12] points will be obtained. It will be shown, via a numerical survey, that there is a loss of global minimality associated with the Darboux point. The foregoing discussion serves to provide an intuitive backdrop, and now we shall examine the preceding notions in some detail.

Multiple Extremal Solutions

Examination of Eq. (10) shows that the roll dynamics are governed solely through the cross-coupling term $q\,r$. For some boundary conditions, it is conceivable that the pitch motion q is dominant in shaping the roll motion, whereas for other boundary conditions, the yaw motion r assumes that responsibility. This dichotomy in the assumed dominant role in governing roll motion yields two basic subfamilies of extremals. The one in which q is more dynamic is labeled to be the q-type extremal, and similarly, if r is more dynamic, the extremal is called the r-type extremal. To exhibit these phenomena and also to establish the nomenclature for the classification of different extremals, we present the following numerical example.

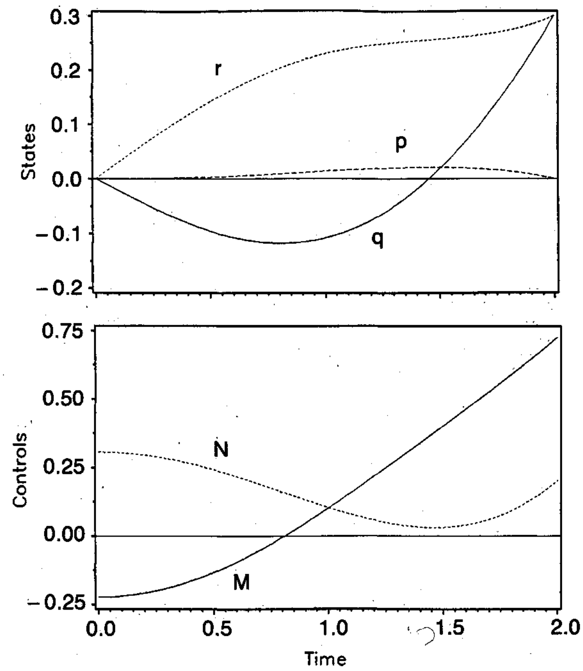

Fig. 1 A q-type extremal; states p, q, and r and controls M and N time histories.

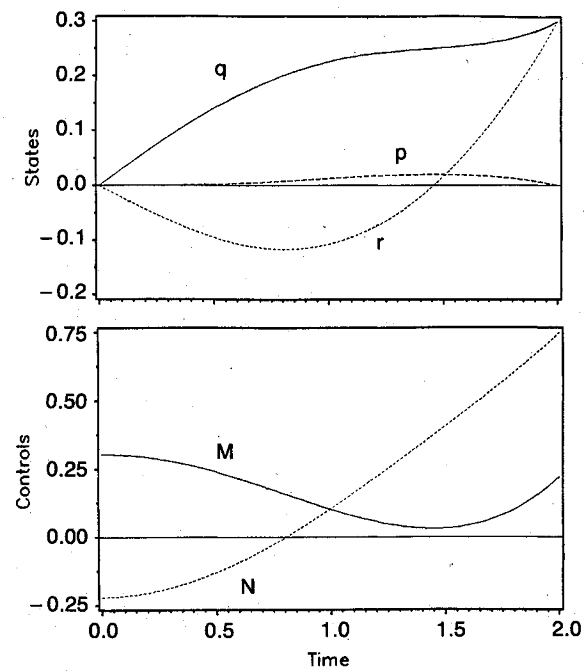

Fig. 2 An r-type extremal; states p, q, and r and controls M and N time histories.

Example 1: For the following initial and final data

$$p_0 = 0 \qquad q_0 = 0 \qquad r_0 = 0$$

$$p_f = 0 \qquad q_f = 0.3 \qquad r_f = 0.3$$

$$t_f = 2$$

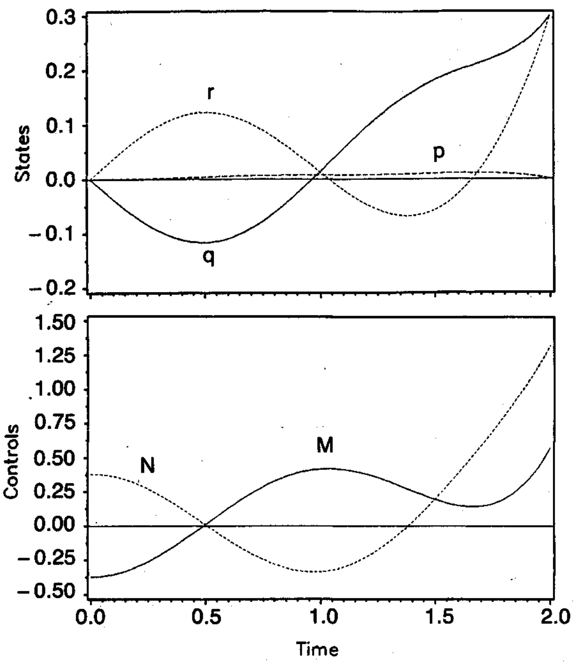

Fig. 3 A qr-type extremal; states p, q, and r and controls M and N time histories.

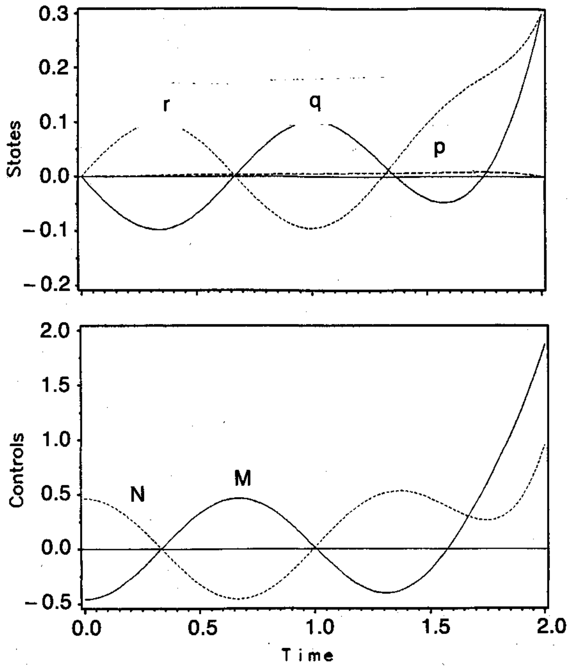

Fig. 5 A qrq-type extremal; states p, q, and r and controls M and N time histories.

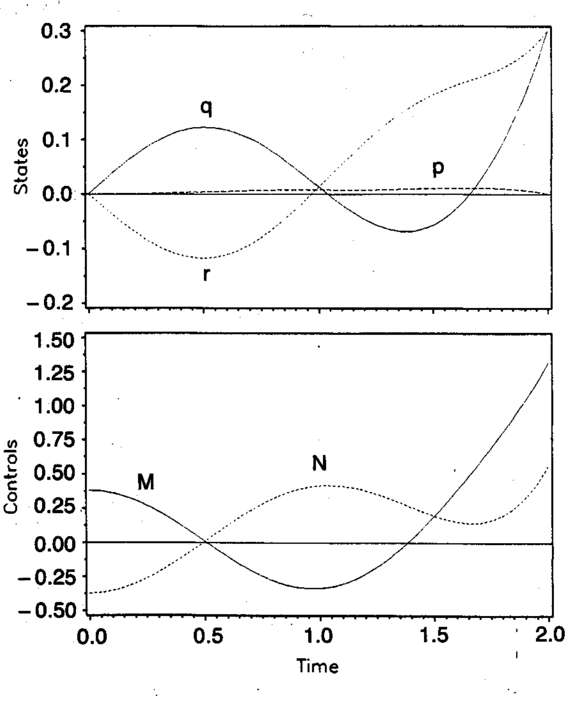

Fig. 4 An rq-type extremal; states p, q, and r and controls M and N time histories.

we obtained the following extremal solutions.

1) Extremal 1 is labeled as a q-type extremal, corresponds to $\lambda_p(0) \simeq -2.45475$, $\lambda_q(0) \simeq 0.22402$, and $\lambda_r(0) \simeq -0.30545$, and has a cost $\simeq 0.14034$. Figure 1 shows the extremal trajectory and the control histories.

2) Extremal 2 is labeled as an r-type extremal, corresponds to $\lambda_p(0) \simeq -2.47985$, $\lambda_q(0) \simeq -0.30264$, and $\lambda_r(0) \simeq 0.22088$, and has a cost $\simeq 0.14344$. Figure 2 shows the extremal trajectory and the control histories.

3) Extremal 3 is labeled as a qr-type extremal, corresponds to $\lambda_p(0) \simeq -9.88796$, $\lambda_q(0) \simeq 0.37214$, and $\lambda_r(0) \simeq -0.37904$, and has a cost $\simeq 0.28333$. Figure 3 shows the extremal trajectory and the control histories.

4) Extremal 4 is labeled as an rq-type extremal, corresponds to $\lambda_p(0) \simeq -9.85119$, $\lambda_q(0) \simeq -0.37996$, and $\lambda_r(0) \simeq 0.37278$, and has a cost $\simeq 0.28216$. Figure 4 shows the extremal trajectory and the control histories.

5) Extremal 5 is labeled as a qrq-type extremal, corresponds to $\lambda_p(0) \simeq -22.19089$, $\lambda_q(0) \simeq 0.46064$, and $\lambda_r(0) \simeq -0.46088$, and has a cost $\simeq 0.42379$. Figure 5 shows the extremal trajectory and the control histories.

Observe that, since the initial and final values of the roll rate p are zero, to achieve such a maneuver requires that the sign of the product qr [Eq. (10)] change on the interval $[0,t_f]$. If one of the variables remains of a single sign, then the other must be more dynamic or control-like, yielding the q- and r-type extremals. Furthermore, the maneuver may be achieved in two stages, although with an increase in cost, where first $q(r)$ then $r(q)$ assumes the role of a control-like variable, yielding extremals labeled to be $qr(rq)$-type extremals. One may extend the same arguments for the more oscillatory solutions, like the qrq-type extremal. For brevity, we do not show these more oscillatory solutions.

Here we digress to mention that angular positions, obtained by simply integrating the kinematic equations along an extremal trajectory, are different for the various extremals presented above. For instance, if we start out straight and level, $\psi_0 = \theta_0 = \phi_0 = 0$ deg, we end up with the following angular orientations:

1) $\psi_f = 22.47$ deg, $\theta_f = -2.10$ deg, and $\phi_f = -0.58$ deg for the q-type extremal.

2) $\psi_f = -1.85$ deg, $\theta_f = 22.02$ deg, and $\phi_f = 2.04$ deg for the r-type extremal.

3) $\psi_f = 5.51$ deg, $\theta_f = 5.45$ deg, and $\phi_f = 0.74$ deg for the qr-type extremal.

4) $\psi_f = 5.48$ deg, $\theta_f = 5.35$ deg, and $\phi_f = 1.16$ deg for the rq-type extremal.

5) $\psi_f = 6.04$ deg, $\theta_f = 1.24$ deg, and $\phi_f = 0.48$ deg for the qrq-type extremal.

However, note that angular position coordinates are not included in the optimal control problem.

Envelopes of Extremal Subfamilies

With a possibility of a multitide of extremal solutions, some of which are shown in example 1, we next investigate the domains of existence in the parameter space of the different types of extremal solution. To get a better grasp on this problem, we fix the values for five of the seven parameters. For $p_0 = q_0 = r_0 = p_f = 0$ and $t_f = 2.0$, we investigate the extremal solutions by varying the other two parameters, namely q_f and r_f. In this reduced parameter space, we pick an extremal solution and carefully deform the boundary conditions q_f and r_f, solving a series of TPBVPs and ensuring that we track the same extremal solution type. For example, we pick the q-type extremal of example 1, fix $r_f = 0.3$, and traverse the boundary conditions along the arc ac (see Fig. 6). It was found that no members of the q-type subfamily could be found beyond the point b. The point b is referred to as the "envelope contact point." We carry out the same procedure for different values of r_f and establish the locus of the envelope contact points for the q-type extremal subfamily, shown as arc Ob in Fig. 6. Above the envelope we have members of the q-type subfamily of extremals, but none below.

Similarly, we pick the r-type extremal of example 1, fix $q_f = 0.3$, and traverse the boundary conditions along the arc ae (see Fig. 6). It was found that no members of the r-type subfamily could be found above the point d. Following the same procedure as above, we establish the locus of the envelope contact points for the r-type extremal subfamily, shown as arc Od in Fig. 6. The broken lines shown in Fig. 6, almost coincident with arcs Od and Ob, are the envelopes of q- and r-type subfamilies as predicted by the approximate problem of Ref. 9.

Locus of Darboux Points

To investigate the comparative cost of the different extremal types, we scan the boundary condition q_f for $r_f = 0.3$ tracking the q-, r-, qr-, etc.-type subfamilies. The results are shown in Fig. 7. From this study it appears the more oscillatory solutions, i.e., the qr-type, rq-type,... and so forth, extremals are more expensive than the q- or r-type extremals. However, between the q- and the r-type extremals, there is competition. For small values of q_f, the q-type extremal is the minimizing solution, whereas for large values of q_f, the r-type extremal is the minimizing solution. The two extremals yield the same cost at the Darboux point. In the reduced parameter space of end conditions q_f and r_f, the locus of Darboux points is obtained, as shown in Fig. 8. The significance of the Darboux locus is that it demarcates this reduced parameter space into two regions, one in which the q-type extremals provide the globally minimal solution and the other where the r-type extremals are globally minimal. A summary of important results is shown in Fig. 8. Arc Ob is the envelope of the r-type extremal subfamily, arc Od is the envelope of the q-type extremal subfamily, and arc Oc is the locus of Darboux points. In the region bOd, both q- and r-type extremals exist, q-type extremals are globally minimal in the region aOc, and r-type extremals are globally minimal in the region cOe.

Further, envelopes of the q- and r-type subfamilies of extremals and the Darboux locus is examined as we change the parameter p_f. For $p_f = 0.1$ and -0.1, the results are shown in Figs. 9 and 10, respectively.

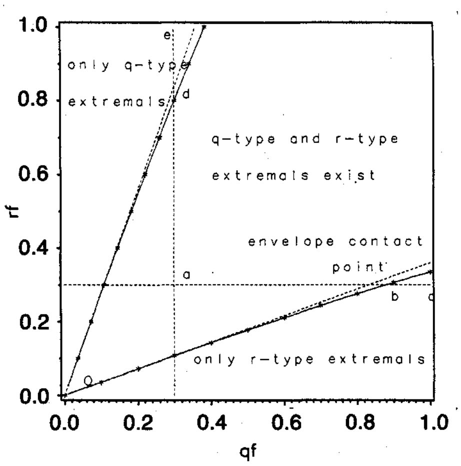

Fig. 6 Domain of existence: q- and r-type extremal subfamilies.

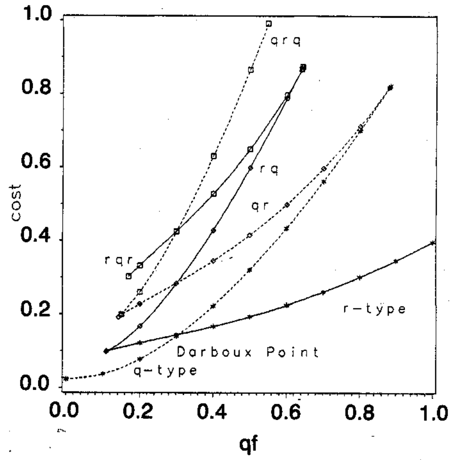

Fig. 7 Cost of different extremal subfamilies: $c(t_f)$ vs q_f for $r_f = 0.3$.

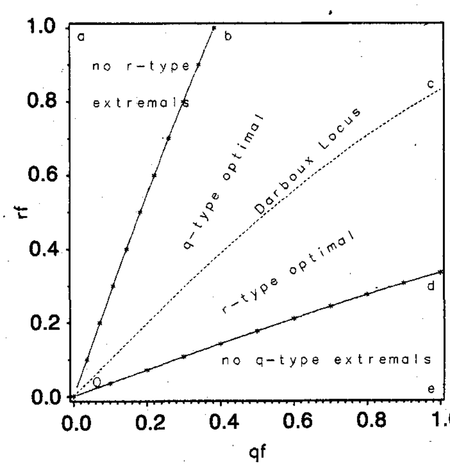

Fig. 8 Locus of Darboux points; global optimality of q- and r-type subfamilies.

Monotonicity of Cost with Maneuver Time

Practical considerations require investigation of the tradeoff between maximum available control effort and maneuver time for given maneuvers. To achieve this, we study the cost, i.e., control effort, as a function of prescribed maneuver time t_f, while holding the initial and final states fixed. As a typical case, we present results obtained for extremals 1 and 2 of example 1 in Fig. 11. For $t_f = 2$, the q-type extremal (extremal 1) yields the globally minimal solution, and the same trend is maintained for all values of t_f. For this extremal solution, we observe that the cost asymptotically approaches infinity as the maneuver time approaches zero and vice versa. The monotone behavior of cost with maneuver time is the key element of the reciprocity argument with which one may obtain minimum time solutions for a given maximum value of control effort by picking the smallest time t_f such that the cost

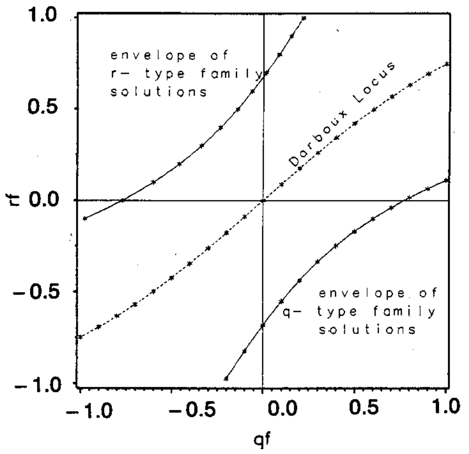

Fig. 9 Darboux locus and domains of q- and r-type subfamilies for $p_f = -0.1$.

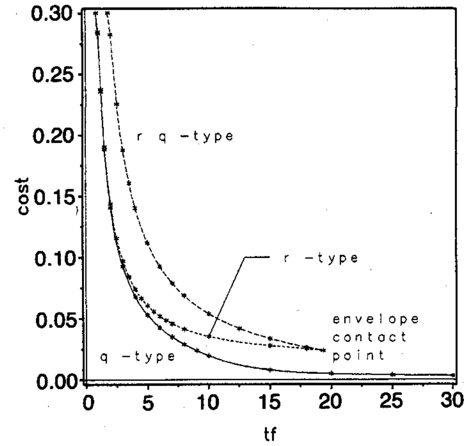

Fig. 11 Cost vs maneuver time tradeoff.

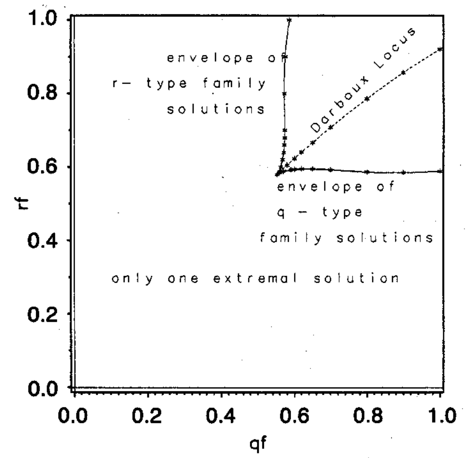

Fig. 10 Darboux locus and domains of q- and r-type subfamilies for $p_f = 0.1$.

does not exceed the prescribed maximum effort. It is interesting to note that the r-type extremal (extremal 2), which is one of the nonminimal extremal solution ceases to exist at $t_f \simeq 19.355$ s. This is the envelope contact point, and no members of the r-type extremal subfamily were found to exist for higher values of t_f.

It is evident that over large domains of boundary conditions, first-order necessary conditions provide several distinct candidate minimizers. It is imperative, therefore, to verify local optimality of these candidate extremals. In the following section, we develop the test for second-order necessary condition, which together with the Legendre-Clebsch condition, provide assurance of the minimizing character (at least locally) of candidate extremals.

V. Second-Order Necessary Conditions—Jacobi Test

In classical calculus of variation problems, the Jacobi necessary condition[13] checks for the minimizing character of a candidate extremal by examining the second variation. The process is achieved via the accessory minimum problem, which seeks to find an admissible variation that offers the most severe competition, in the sense of minimizing the second variation. If no such nonzero system of variations can be found for a test extremal without corners, then the extremal furnishes a relative minimum. Similar arguments have been extended to the more general nonsingular problems of optimal control by several investigators, notably Breakwell and Ho,[14] Moyer,[15] and Kelley and Moyer.[16]

Recall that the rigid-body rotation problem defined in Sec. III is regular, i.e., all its extremals satisfy the convexity condition. Therefore, if a test extremal satisfies the second-order necessary condition, it provides a local minimum. The accessory minimum problem for rigid-body rotation problem defined by Eqs. (6–9) is obtained using the procedure outlined in Ref. 16. The equations of variation obtained by linearizing the state and adjoint system [Eqs. (10–13) and Eqs. (21–24)] and the optimality conditions [Eqs. (19) and (20)] about a test extremal are

$$\dot{\delta p} = -r\delta q - q\delta r \tag{26a}$$

$$\dot{\delta q} = r\delta p + p\delta r + \delta M \tag{26b}$$

$$\dot{\delta r} = -q\delta p - p\delta q + \delta N \tag{26c}$$

$$\dot{\delta c} = M\delta M + N\delta N \tag{26d}$$

We seek to minimize the second variation

$$\delta^2 \mathcal{J} = \frac{1}{2}\int_0^{t_f} [2(-\lambda_p \delta q \delta r + \lambda_q \delta p \delta r - \lambda_r \delta p \delta q) + \lambda_0(\delta M^2 + \delta N^2)]\, \mathrm{d}t \tag{27}$$

over all possible admissible variations. The Euler-Lagrange equations for the accessory minimum problem are obtained to be

$$\delta\dot{\lambda}_p = -\lambda_q \delta r + \lambda_r \delta q - r\delta\lambda_q + q\delta\lambda_r \tag{28a}$$

$$\delta\dot{\lambda}_q = \lambda_p \delta r + \lambda_r \delta p + r\delta\lambda_p + p\delta\lambda_r \tag{28b}$$

$$\delta\dot{\lambda}_r = \lambda_p \delta q - \lambda_q \delta p + q\delta\lambda_p - p\delta\lambda_q \tag{28c}$$

$$\delta\dot{\lambda}_0 = 0 \tag{28d}$$

while the minimum principle for the accessory minimum problem gives

$$\delta M = -\frac{1}{\lambda_0}(\delta\lambda_q + M\delta\lambda_0) \tag{29a}$$

$$\delta N = -\frac{1}{\lambda_0}(\delta\lambda_r + N\delta\lambda_0) \tag{29b}$$

Since we are seeking competitors of the test extremal, the variation, if admissible, must satisfy the boundary conditions of the test extremal, i.e.,

$$\delta p(0) = 0 \qquad \delta q(0) = 0$$
$$\delta r(0) = 0 \qquad \delta c(0) = 0 \tag{30a}$$

and

$$\delta p(t_f) = 0 \qquad \delta q(t_f) = 0$$
$$\delta r(t_f) = 0 \qquad \delta\lambda_0(t_f) = 0 \tag{30b}$$

Here we digress to mention that there exists a first integral

$$\frac{\mathrm{d}}{\mathrm{d}t}(\lambda_p\delta p + \lambda_q\delta q + \lambda_r\delta r + \lambda_0\delta c) = 0$$

or

$$\lambda_p\delta p + \lambda_q\delta q + \lambda_r\delta r + \lambda_0\delta c = \text{const} \tag{31}$$

which may be verified by differentiation and using the relevant equations. Making use of the boundary condition (30a), we can evaluate the constant to be zero. Further, if we evaluate Eq. (31) at t_f, we get

$$\lambda_0(t_f)\delta c(t_f) = 0 \tag{32}$$

Since $\lambda_0(t_f)$ was prescribed to be unity (using the normality assumption), $\delta c(t_f)$ must be zero.

Equation (31) has an interesting geometric interpretation. If we think about the locus of endpoints obtained by extremals originating from a fixed point in the state space (the origin in our case), and stopped at fixed final time, the three-dimensional manifold defines an isochronal hypersurface, or wavefront.[15] The variations $\delta x^T = (\delta p, \delta q, \delta r, \delta c)$ lie in the tangent manifold of the wavefront. Thus the adjoint vector must be normal to the wavefront.

Equations (26) and (28) are a set of linear, although time-varying, differential equations, which along with the optimality conditions (29), and the boundary conditions defined by Eq. (30) constitute a two-point boundary-value problem. Note that the trivial solution satisfies all the necessary conditions and provides the minimum value for $\delta^2\mathscr{J}$ equal to zero. Furthermore, zero is the global minimum of $\delta^2\mathscr{J}$, as can be verified by straightforward integration of Eq. (27) using the Euler-Lagrange equations for the accessory minimum problem. Therefore, our search for competition reduces to finding a nontrivial solution of the aforementioned linear two-point boundary-value problem. However, we must specifically exclude those solutions that amount to a mere scaling of the adjoint vector of the test extremal.

Owing to the linearity and homogeneity of the state and adjoint variations, we may express the state variation in terms of the unspecified initial values of the adjoint variations as follows:

$$\begin{bmatrix} \delta p(t) \\ \delta q(t) \\ \delta r(t) \\ \delta\lambda_0(t) \end{bmatrix} = \begin{bmatrix} \frac{\partial p}{\partial\lambda_{p_0}} & \frac{\partial p}{\partial\lambda_{q_0}} & \frac{\partial p}{\partial\lambda_{r_0}} & \frac{\partial p}{\partial\lambda_{0_0}} \\ \frac{\partial q}{\partial\lambda_{p_0}} & \frac{\partial q}{\partial\lambda_{q_0}} & \frac{\partial q}{\partial\lambda_{r_0}} & \frac{\partial q}{\partial\lambda_{0_0}} \\ \frac{\partial r}{\partial\lambda_{p_0}} & \frac{\partial r}{\partial\lambda_{q_0}} & \frac{\partial r}{\partial\lambda_{r_0}} & \frac{\partial r}{\partial\lambda_{0_0}} \\ \frac{\partial\lambda_0}{\partial\lambda_{p_0}} & \frac{\partial\lambda_0}{\partial\lambda_{q_0}} & \frac{\partial\lambda_0}{\partial\lambda_{r_0}} & \frac{\partial\lambda_0}{\partial\lambda_{0_0}} \end{bmatrix} \begin{bmatrix} \delta\lambda_p(0) \\ \delta\lambda_q(0) \\ \delta\lambda_r(0) \\ \delta\lambda_0(0) \end{bmatrix} \tag{33}$$

If for any $t_c \in (0,t_f]$, we can find a nonzero set of initial adjoint variations such that the column vector on the left is zero, then a nontrivial solution of the accessory minimum problem is possible. The time t_c is then the conjugate time. For the purposes of locating the conjugate time, we may use the vanishing of the determinant of the test matrix as the criteria. Further, the last component amounts to scaling of the multiplier $\lambda_0 + \delta\lambda_0$. Therefore, we may delete the last row and column of the preceding matrix. The question of whether or not a particular extremal is locally optimal now reduces to studying whether the following submatrix has a drop in rank on the interval $(0,t_f]$.

$$\Phi(t) = \begin{bmatrix} \frac{\partial p}{\partial\lambda_{p_0}} & \frac{\partial p}{\partial\lambda_{q_0}} & \frac{\partial p}{\partial\lambda_{r_0}} \\ \frac{\partial q}{\partial\lambda_{p_0}} & \frac{\partial q}{\partial\lambda_{q_0}} & \frac{\partial q}{\partial\lambda_{r_0}} \\ \frac{\partial r}{\partial\lambda_{p_0}} & \frac{\partial r}{\partial\lambda_{q_0}} & \frac{\partial r}{\partial\lambda_{r_0}} \end{bmatrix} \tag{34}$$

The elements of the test matrix $\Phi(t)$ or the sensitivity of the states with respect to the initial adjoints can be obtained numerically using the following scheme. Four sets of initial conditions are assigned to the differential system of Eqs. (26) and (28) with the algebraic condition of Eq. (29) appropriately introduced. The initial state variations $\delta x(0)^T = [\delta p(0)\delta q(0)\delta r(0)\delta c(0)]$ for each set is zero, while the initial adjoint variations $\delta\lambda(0)^T = [\delta\lambda_p(0)\delta\lambda_q(0)\delta\lambda_r(0)\delta\lambda_0(0)]$ for the ith set consists of all zero except the ith element of $\delta\lambda(0)^T$, which is taken to be unity. The state variations $\delta x_i(t)^T$ obtained by integrating this differential system along a test extremal provides the sensitivity of the states with respect to $\delta\lambda_{i0}$, and thus $\Phi(t)$ can be computed. If $\Phi(t)$ has full rank, or alternatively if $[\Phi(t) \neq 0 \forall t \in (0,t_f)]$, then the extremal passes the second-order test successfully, and the test extremal does indeed furnish a relative minimum. If $|\Phi(t_c)| = 0$ for some $t_c \in (0,t_f]$, then neighboring competition exists, and the local optimality of the test extremal is in question.

The major result of the investigation of the second-order necessary conditions is that all extremals that belong to the q- and r-type subfamilies of extremals, in general, satisfied the second-order necessary conditions. If we deform the boundary conditions in a manner that members of these subfamilies of extremals contact their envelopes (see Fig. 6, for example), then at envelope contact points, the second-order necessary conditions are satisfied only in their weakened form, i.e., there is a conjugate point at t_f. This particular feature is reminiscent of the catenary-type extremals of surfaces of revolution of minimum area.[17] Furthermore, the q- and r-type subfamilies of extremals exhibit the popularly believed notion that the point at which an extremal ceases to be globally optimal does not, as a rule, coincide with envelope contact but rather precedes it. In other words, global optimality is lost before local optimality, as is clearly evident from Fig. 8.

The extremal solutions that are oscillatory in character, labeled as qr-type, rq-type, qrq-type extremals, and so forth, were found to fail the abovementioned rank test. Thus all these extremal types, although satisfying the stationarity conditions, fail the second-order necessary conditions, which provide the guarantee of minimizing character. Further, it was found that all extremals belonging to the qr-type or rq-type subfamilies had one conjugate point; extremals belonging to qrq-type or rqr-type subfamilies had two conjugate points and so forth.

VI. Concluding Remarks

Optimal rigid-body angular motions are studied in the absence of direct control over roll motion—a problem inspired in the context of supermaneuverable aircraft. While over a large range of boundary conditions several extremal subfamilies satisfy the stationarity conditions, the more oscillatory-type subfamilies of extremal solutions may be discounted because they do not satisfy the second-order necessary condition. Thus, our choice for minimizing solution narrows down to two basic subfamilies, classified as the q- and r-type subfamilies of extremals. These two basic subfamilies arise due to a dichotomy in the assumed dominant role of state variables q or r in shaping the indirectly controlled state p. Envelopes of these two subfamilies are obtained in a reduced parameter space. A locus of Darboux points is obtained, which is found to demarcate the domains over which the two subfamilies were globally minimal.

As an outcome of the present study, the authors feel that the problem of optimal control of rigid-body angular motions, wherein the gyroscopic terms have influence in shaping the extremal trajectory and are not overwhelmed by controlling moments, are to be treated with extreme caution. While the present study deals with an extreme case where the roll dynamics are solely governed by the gyroscopic cross-coupling term, instances of multiple extremal solutions are found with a limited amount of roll control. As a final note, we add that future studies of optimal flight of supermaneuverable aircraft including rigid-body rotational dynamics need to consider the issues discussed herein. Recognizing, however, the importance of these issues depends on the flight conditions and the extent of aerodynamic influence.

Acknowledgment

This work was supported in part by the Defense Advanced Research Projects Agency under contract F49620-87-C-0016.

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